Capital Asset Pricing Model

I. CAPM: Assumptions and Prediction

1. Perfect markets
   - Perfect competition: Each investor has no effect on prices or returns
   - No taxes
   - No transaction costs
   - All assets are traded and perfectly divisible
   - No short sale constraints
   - Borrow and lend at the same risk-free rate, total borrowing = total lending

2. Investors only worry about mean and variance of end-of-period wealth

3. Homogeneous expectations

Under these assumptions, CAPM says that the return of any Stock *i*, r_i, should satisfy

$$E[r_i] = r_f + \beta_i \left[E(r_M) - r_f \right]$$

Where r_f is the risk free rate, r_M is the market portfolio return, and

$$\beta_i = \frac{Cov(r_i, r_M)}{Var(r_M)}$$

II. CAPM Market Portfolio and Risk-Free Rate

A. How do we know that everyone would invest only in same portfolio?

B. How do we know that the market will be "cleared"—that everyone gets to invest however much they want in the efficient portfolio?

Notice that if some investors become less risk-averse, the market portfolio becomes more risky.

Note: Under the assumption of CAPM, the market portfolio should include all wealth. However, we typically use a portfolio of exchange-traded stocks. For example, we might use the value-weighted index of NYSE stocks

III. Beta Pricing

A. Where does the beta pricing formula come from?

The slope of the capital allocation line tells us the additional return an investor will get if he takes up one standard-deviation of additional risk by buying . By the two-point method of obtaining slope, this is

$$\frac{dr_P}{d\sigma_P} = \frac{r_M - r_f}{\sigma_M - \sigma_f} = \frac{r_M - r_f}{\sigma_M}$$

Now suppose the investor decided to invest a little less in the market portfolio and more in a particular asset i, what will the tradeoff between portfolio return and risk become?

If the investor keeps a fraction α of his wealth in the market portfolio and invest the remaining $1 - \alpha$ in asset i, this new portfolio Q will have

$$r_Q = \alpha r_M + (1-\alpha) r_i$$

$$\sigma_Q = \{Var[\alpha r_M + (1-\alpha) r_i]\}^{0.5}$$

$$= \{\alpha^2 \sigma_M^2 + (1-\alpha)^2 \sigma_i^2 + 2\alpha(1-\alpha) Cov(r_M, r_i)\}^{0.5}$$

Taking derivatives what respect to α ,

$$\frac{dr_Q}{d\alpha} = r_M - r_i$$

$$\frac{d\sigma_Q}{d\alpha} = 0.5\{\sigma_Q^2\}^{-0.5}\{2\alpha\sigma_M^2 - 2(1-\alpha)\sigma_i^2 + 2[(1-\alpha) - \alpha]Cov(r_M, r_i)\}$$

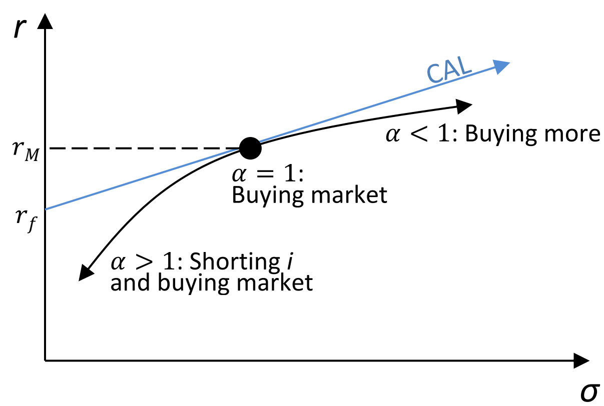
$$= \sigma_Q^{-1}[\alpha\sigma_M^2 - (1-\alpha)\sigma_i^2 + (1-2\alpha)Cov(r_M, r_i)]$$

Dividing the first equation by the second gives us the tradeoff between return and risk as we deviate,

$$\frac{dr_Q}{d\sigma_Q} = \frac{dr_Q/d\alpha}{d\sigma_Q/d\alpha} = \frac{r_M - r_i}{\sigma_Q^{-1}[\alpha\sigma_M^2 - (1-\alpha)\sigma_i^2 + (1-2\alpha)Cov(r_M, r_i)]}$$

When $\alpha = 1$, the investor is holding exactly the market portfolio, so $r_Q = r_M$ and $\sigma_Q = \sigma_M$.

Furthermore, as argued before, deviating to any asset other than the risk-free asset will result in a worse return-risk tradeoff. Plotting this graphically (for illustration, the case when i is more risky than the market portfolio),



It is apparent from the diagram that

$$\left.\frac{dr_Q}{d\sigma_Q}\right|_{\alpha=1} = \frac{dr_P}{d\sigma_P} = \frac{r_M - r_f}{\sigma_M}$$

Which means

$$\left.\frac{r_M - r_i}{\sigma_Q^{-1}[\alpha\sigma_M^2 - (1-\alpha)\sigma_i^2 + (1-2\alpha)Cov(r_M, r_i)]}\right|_{\alpha=1} = \frac{r_M - r_f}{\sigma_M}$$

$$\frac{r_M - r_i}{\sigma_M^{-1}[\sigma_M^2 - Cov(r_M, r_i)]} = \frac{r_M - r_f}{\sigma_M}$$

$$r_i = r_f + \frac{Cov(r_M, r_i)}{\sigma_M^2}(r_M - r_f)$$

B. What does the CAPM Beta formula says?

1. β measures each asset's contribution to the variance of the market portfolio, which we know is the optimal portfolio. (You will be asked to derive this property of β in Assignment 1.)

2. Investors only care about the risk and expected return of their *optimal portfolios*. Therefore, they are only concerned with the impact of an additional asset on the risk and return of their portfolio.

3. Since the only relevant risk of an asset is its *marginal contribution to the risk of the market portfolio*, this is the only risk that gets compensated.

   For an asset that goes against the market, its return should be negative! Why?