Static Portfolio Optimization

I. Common Utility Functions

The most commonly used utility functions in financial economics belong to the hyperbolic absolute risk aversion (HARA) family.

$$u(x) = \tau \cdot \left(\mu + \frac{x}{\gamma}\right)^{1-\gamma}$$

Where τ, μ, γ are some numbers. Looks ugly doesn't it? Fortunately, we are only focusing on three special cases.

i. Quadratic Utility

$$u(x) = ax^2 + bx + c$$

Special property:

ii. Constant Absolute Risk Aversion (CARA)

$$u(x) = \frac{-e^{-Ax}}{A}$$

Where *A* is rate of absolute risk aversion.

iii. Constant Relative Risk Aversion (CRRA)

$$u(x) = \frac{x^{1-R}}{1-R}$$

Where *R* is rate of relative risk aversion.

What happens when R = 1? $u(x) = \ln(x)$.[1]

[1] Note that the limit of $\frac{x^{1-R}}{1-R}$ is *not* $\ln(x)$. To get from the former to the latter, we need to first add $-\frac{1}{1-R}$, which is alright because this is a monotonic transformation that preserves the preference ordering, and then apply L'Hospital's Rule.

II. Single Risky Asset Optimization

Let's consider the case where they are only two assets in this world—a risk-free, interesting bearing one, and a risky one. As you shall find out later on, this is in fact not a bad assumption. For simplicity, we shall call the riskless asset "bond" and the risky asset "stock". Bond will have a sure return of r, while stock will have a risky return of $r + s$. Notice how we express the return of stock as the return of bond plus a number. In finance s is called *excess return*.

Let us give our hypothetical investor a wealth of $\$w$, and suppose this investor invest $\$\theta$ of her wealth in stock. The her final wealth is

$$\boxed{(w - \theta)r + \theta(r + s) = wr + \theta s}$$

Remember from your immediate microeconomics course, we maximize a utility function by taking its first derivative:

$$\boxed{\frac{d}{d\theta} U(wr + \theta s) = 0}$$

Simple? Unfortunately depending on how s looks like, the math is not necessarily easy. So as economists, we do what we always do—let's assume how s looks like!

<u>Case ii. CARA</u>

Assume s is *normally distributed* with mean μ and standard deviation σ.
What does a normal curve looks like?

Properties: Let a be any number, $s + a$ is normal, $s * a$ is normal

Second concept: e^s is *log-normally distributed*.

What does log-normal means?

What does a log-normal curve looks like?

Properties: Let a be any number, $e^s * a$ is log-normal, $e^s + a$ is *shifted* log-normal

Actual Math

ONE RULE TO REMEMBER: If x is *log-normal*

$$\log(E[x]) = E[\log(x)] + \frac{1}{2}Var[\log(x)]$$

Example: $u(x) = \frac{-e^{-4x}}{4}$

The optimal dollar amount to invest in stock is given by

$$\begin{aligned}
\underset{\theta}{\operatorname{argmax}}\{E[u(wr+\theta s)]\} &= \operatorname{argmax}\left\{E\left[\frac{-e^{-4(wr+\theta s)}}{4}\right]\right\} \\
&= \operatorname{argmax}\left\{-\frac{1}{4}E\left[e^{-4(wr+\theta s)}\right]\right\} \\
&= \operatorname{argmin}\left\{\frac{1}{4}E\left[e^{-4(wr+\theta s)}\right]\right\} \\
&= \operatorname{argmin}\left\{E\left[e^{-4(wr+\theta s)}\right]\right\} \\
&= \operatorname{argmin}\left\{\log\left(E\left[e^{-4(wr+\theta s)}\right]\right)\right\} \\
&= \operatorname{argmin}\left\{E\left[\log\left(e^{-4(wr+\theta s)}\right)\right) + \frac{1}{2}Var\left[\log\left(e^{-4(wr+\theta s)}\right)\right]\right\} \\
&= \operatorname{argmin}\left\{E[-4(wr+\theta s)] + \frac{1}{2}Var[-4(wr+\theta s)]\right\} \\
&= \operatorname{argmin}\left\{-4wr - 4\theta E[s] + \frac{1}{2}\cdot 16\theta^2 Var[s]\right\}
\end{aligned}$$

Taking first derivative to solve the minimization problem:

$$\begin{aligned}
\frac{\partial E[U]}{\partial \theta} &= -4E[s] + 8\cdot 2\theta Var[s] = 0 \\
\theta &= \frac{E[s]}{4\cdot Var[s]} \\
&= \frac{\mu}{4\sigma^2}
\end{aligned}$$

III. Should we invest at all?

Notice from the solutions above. The investors always invest. This result actually holds for any utility function. Intuitively, why?

Why might we think this is not realistic?