1. Leonardo enjoys smoking cigarettes once in a while on his balcony. At this very moment Leonardo has to choose between stopping for a smoke (x) and having tea (y). His utility function is $U(x,y) = x^{\alpha} y^{\beta}$ where α and β are constants. Cigarettes cost p_x each while tea costs p_y per bag. .

   a. Verify that Leonardo's marginal utility of x is $\alpha x^{\alpha-1} y^{\beta}$ and marginal utility of y is $x^{\alpha} \beta y^{\beta-1}$. Solve for Leonardo's optimal combination of cigarettes and tea.

$$\frac{\partial U}{\partial x} = \alpha x^{\alpha-1} y^{\beta}$$

$$\frac{\partial U}{\partial y} = \beta x^{\alpha} y^{\beta-1}$$

$$MRS = \frac{\partial U}{\partial x} \Big/ \frac{\partial U}{\partial y} = \frac{\alpha x^{\alpha-1} y^{\beta}}{\beta x^{\alpha} y^{\beta-1}} = \frac{\alpha y}{\beta x}$$

$$MRS = \text{Price Ratio}$$

$$\frac{\alpha y}{\beta x} = \frac{p_x}{p_y}$$

$$y = \frac{p_x \beta}{p_y \alpha} x$$

$$p_x \cdot x + p_y \cdot y = I$$

$$p_x \cdot x + p_y \cdot \frac{p_x \beta}{p_y \alpha} x = I$$

$$p_x \cdot x + \frac{p_x \beta}{\alpha} x = I$$

$$x = \frac{1}{1+\beta/\alpha} \cdot \frac{I}{p_x}$$

$$y = \frac{p_x \beta}{p_y \alpha} \cdot x = \frac{p_x \beta}{p_y \alpha} \cdot \frac{I}{p_x\left(1+\frac{\beta}{\alpha}\right)} = \frac{\beta/\alpha}{1+\beta/\alpha} \cdot \frac{I}{p_y}$$

b. Suppose the prices of cigarettes and tea bags are both \$1. When will Leonardo consume more cigarettes than tea bags? (In other words, when will $x > y$?)

$$x > y$$

$$\frac{1}{1+\beta/\alpha}\cdot\frac{I}{p_x} > \frac{\beta/\alpha}{1+\beta/\alpha}\cdot\frac{I}{p_y}$$

$$\frac{1}{1+\beta/\alpha}\cdot I > \frac{\beta/\alpha}{1+\beta/\alpha}\cdot I$$

$$\frac{1}{1+\beta/\alpha} > \frac{\beta/\alpha}{1+\beta/\alpha}$$

$$1 > \beta/\alpha$$

$$\alpha > \beta$$

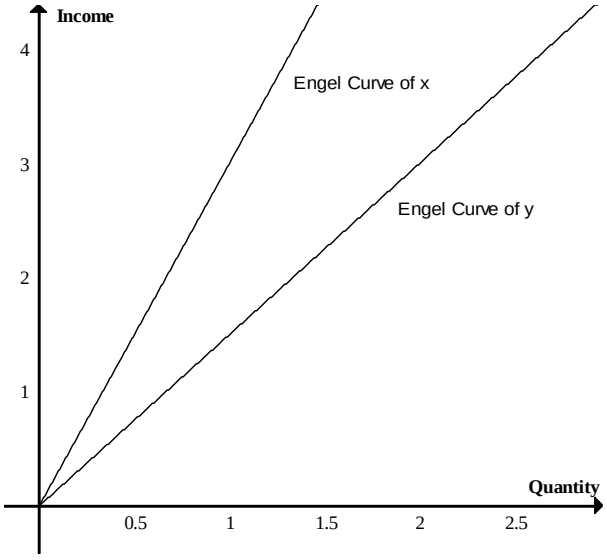
In words, $x > y$ when x's power is greater than y's.

From now on assume $\alpha = 1$ and $\beta = 2$.

c. Maintaining the assumption that the prices of cigarettes and tea bags are \$1, derive Leonardo's Engel Curves for cigarettes and tea bags. Sketch (and I really mean *sketch*) the two curves on a diagram.

$$x = \frac{1}{1+\beta/\alpha}\cdot\frac{I}{p_x} = \frac{I}{3}$$

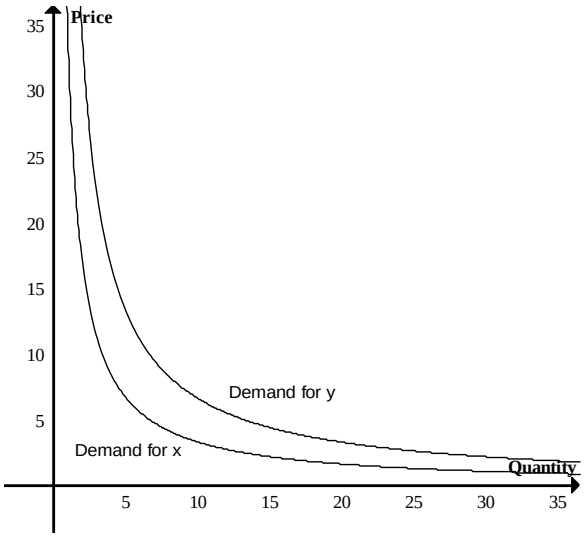
$$y = \frac{\beta/\alpha}{1+\beta/\alpha}\cdot\frac{I}{p_y} = \frac{2I}{3}$$



d. Drop the assumption on prices and assume instead that Leonardo has \$100 in his pocket. Derive the Leonardo's demand for cigarettes and tea. Sketch the two curves on another diagram.

$$x = \frac{1}{1+\beta/\alpha}\cdot\frac{I}{p_x} = \frac{100}{3p_x}$$

$$y = \frac{\beta/\alpha}{1+\beta/\alpha}\cdot\frac{I}{p_y} = \frac{200}{3p_y}$$



e. What are the cross-price elasticities of demand? Does this result depend on the values of Leonardo's income, α and β?
Recall

$$x = \frac{1}{1+\beta/\alpha}\cdot\frac{I}{p_x}$$

$$y = \frac{\beta/\alpha}{1+\beta/\alpha}\cdot\frac{I}{p_y}$$

The demand for x does not depend on the price of y, nor does the demand for y depend on the price of x; so cross-price elasticity is zero for both goods. Mathematically,

$$\frac{\partial x}{\partial p_y} = \frac{\partial}{\partial p_y}\left(\frac{1}{1+\beta/\alpha}\cdot\frac{I}{p_x}\right) = 0$$

$$\frac{\partial y}{\partial p_x} = \frac{\partial}{\partial p_y}\left(\frac{\beta/\alpha}{1+\beta/\alpha}\cdot\frac{I}{p_y}\right) = 0$$

so

$$\varepsilon_{x,p_y} = \frac{p_y}{x}\cdot\frac{\partial x}{\partial p_y} = \frac{p_y}{x}\cdot 0 = 0$$

$$\varepsilon_{y,p_x} = \frac{p_x}{y}\cdot\frac{\partial y}{\partial p_x} = \frac{p_x}{y}\cdot 0 = 0$$

This result is a property of Cobb-Douglas utility function and does not depend on the values of I, α and β.

Deeply disturbed by rising health cost, Governor Arnold decided that a per-unit tax of $1 should be imposed on cigarettes.

f. Arnold does not want to upset the smokers; he believes the state government should issue allowance to all smokers to offset the tax increase. Suppose Arnold has telepathic powers so that he knows the utility function of smokers. How much allowance on income has to be made—in other words, by how much income has to be raised—so that Leonardo gets just as much utility as before?
This part is similar to Problem Set 1 Part III Q.3 Part d.
Assume I = 100, Px = 1 and Py = 1 originally; utility in this case is

$$\begin{aligned} U(x,y) &= U\left(\frac{100}{3},\frac{200}{3}\right) \\ &= \left(\frac{100}{3}\right)^1\cdot\left(\frac{200}{3}\right)^2 \\ &= \frac{4000000}{27} \\ &= 148148.1481 \end{aligned}$$

Now Px = 2. Let S be the subsidy,

$$x = \frac{1}{1+\beta/\alpha}\cdot\frac{I+S}{p_x} = \frac{100+S}{3\cdot 2} = \frac{100+S}{6}$$

$$y = \frac{\beta/\alpha}{1+\beta/\alpha}\cdot\frac{I+S}{p_y} = \frac{2(100+S)}{3\cdot 1} = \frac{2(100+S)}{3}$$

Utility level has to be maintained, so

$$U\left(\frac{100+S}{6},\frac{2(100+S)}{3}\right) = \frac{4000000}{27}$$

$$\frac{100+S}{6}\left[\frac{2(100+S)}{3}\right]^2 = \frac{4000000}{27}$$

$$\frac{4(100+S)^3}{54} = \frac{4000000}{27}$$

$$S = 25.9921 \approx 26$$

g. Does Leonardo smoke as much as before? If not, what has caused him to change his behavior? What do we call this effect in economics?
New cigarette consumption is

$$x = \frac{1}{1+\beta/\alpha}\cdot\frac{I+S}{p_x} = \frac{100+26}{3\cdot 2} = 21 < \frac{100}{3}$$

Leonardo now smokes less. He is substituting a portion of his original cigarette consumption with tea because the former has become more expensive relative to the later. This is called substitution effect in economics.

h. Despite Arnold's effort, the state legislature has voted down the issuance of allowance to smokers. How much cigarettes would Leonardo smoke now? What have caused this further change in behavior and what do we call this effect in economics?
New cigarette consumption is now

$$x = \frac{1}{1+\beta/\alpha}\cdot\frac{I+S}{p_x} = \frac{100+0}{3\cdot 2} = \frac{100}{6} = 16.6667 < 21$$

Leonardo now smokes even less. Compared to part g he has lower real-purchasing power—measured in highest utility achievable—because of the price increase and lack of subsidy. This is called income effect in economics.

i. Forget the utility function given above for a moment, is it possible that Leonardo smokes more than before with the tax in place? Illustrate this with an indifference-curve-budget-constraint diagram.
Any standard diagram for a Giffen good.

j. In reality Arnold does not have telepathic capacities; instead he gives enough allowance to Leonardo such that he could just afford what he had been purchasing before the tax. Will this allowance be higher or lower than that in part f.? Why is there such as difference?
To purchase the same amount of cigarettes and tea as before the tax requires a subsidy of

$$100 + S = 2 \cdot \frac{100}{3} + 1 \cdot \frac{200}{3}$$

$$S = \frac{100}{3}$$

$$> 26$$

So a higher subsidy is paid compared to part f. The difference stems from the ignorance of the substitution effect—that Leonardo can mitigate the effect of the tax by substituting cigarettes with tea.