First Price Auction Example

2 risk neutral bidders. Valuations uniformly distributed on [0,10]

Since the two bidders are risk neutral—use this assumption unless the question state otherwise—they maximizes expected payoff.

For any given bid b_1 bidder 1's expected payoff is given by
Expected Payoff with b_1 *= (probability of winning with* b_1*) * (winning payoff with* b_1*) + (probability of losing with* b_1*) * (losing payoff with* b_1*)*

Winning gives $v_1 - b_1$, where v_1 is bidder 1's valuation of the good in auction. Losing gives 0. So,

$$\begin{aligned} E[X|b_1] &= \Pr(winning|b_1)\cdot(v_1 - b_1) + \Pr(losing|b_1)\cdot 0 \\ &= \Pr(winning|b_1)\cdot(v_1 - b_1) \end{aligned}$$

We need to figure out what $\Pr(winning \mid b_1)$ is. We know bidder 1 wins whenever $b_1 > b_2$. Here we have to do something that mathematicians often do: we *guess* that $b_2 = c \cdot v_2$. How are we suppose to come up with the guess? Well...it's a guess. In any case, with the guess what we have is bidder 1 wins whenever $b_1 > c \cdot v_2$. So

$$\Pr(winning \mid b_1) = \Pr(b_2 < b_1) = \Pr(c \cdot v_2 < b_1) = \Pr\left(v_2 < \frac{b_1}{c}\right)$$

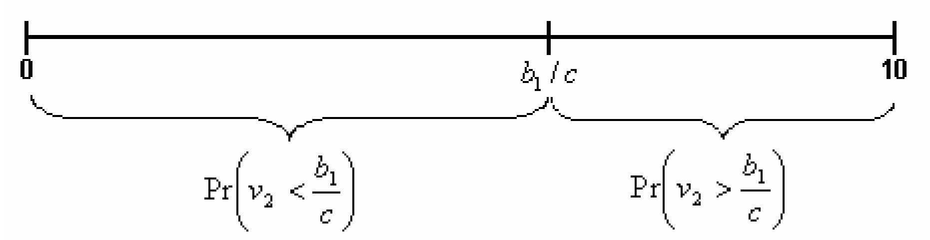
This is where uniform distribution comes into play. If a number *x* is drawn from a uniform distribution of [*a*,*b*], the probability that *x* is smaller than any number *c*, $a \le c \le b$ is

$$\frac{c-a}{b-a}$$

Therefore

$$\Pr(winning \mid b_1) = \Pr\left(v_2 < \frac{b_1}{c}\right) = \frac{b_1/c - 0}{10 - 0} = \frac{b_1}{10c}$$

We can represent this result graphically on a line with a length equal to the range of the uniform distribution:



Putting this result back into the expected payoff formula we have above,

$$E[X \mid b_1] = \frac{b_1}{10c} \cdot (v_1 - b_1)$$

To find the b_1 that maximizes expected payoff we differentiate the above formula with respect to b_1,

$$\frac{\partial E[X \mid b_1]}{\partial b_1} = 0$$

$$\frac{\partial}{\partial b_1}\left[\frac{b_1}{10c} \cdot (v_1 - b_1)\right] = 0$$

$$\frac{v_1}{10c} - \frac{2b_1}{10c} = 0$$

$$b_1 = \frac{v_1}{2}$$

So there we go—the optimal bid for bidder 1 is half her valuation. Note that this optimal bidding strategy does not depend on bidder 2's valuation or bid at all. You might have guessed that bidder 1's bid should increase with bidder 2's bid; remember that, however, raising b_1 also lowers the winning payoff. With uniform distribution the two effects happen to cancel each other.

What about bidder 2? Since the two bidders are identical in every way, we only need to work out the solution for bidder 1. But that's not all, for the solution we have confirms that indeed the optimal bid for any bidder *should* be in the form of $b_i = c \cdot v_i$; so our "guess" is correct after all.

The calculation goes along the same direction when there are more than two bidders; with *n* bidders, the probability of winning is

$$\begin{aligned} \Pr(winning \mid b_1) &= \Pr(b_2, \ldots, b_n < b_1) \\ &= \Pr\left(v_2, \ldots, v_n < \frac{b_1}{c}\right) \\ &= \Pr\left(v_2 < \frac{b_1}{c}\right) \cdot \ldots \cdot \Pr\left(v_n < \frac{b_1}{c}\right) \\ &= \left(\frac{b_1}{10c}\right)^{n-1} \end{aligned}$$

Plug this back into the expected utility formula and maximize,

$$\frac{\partial E[X \mid b_1]}{\partial b_1} = 0$$

$$\frac{\partial}{\partial b_1}\left[b_1^{n-1} \cdot (v_1 - b_1)\right] = 0$$

$$(n-1)b_1^{n-2}v_1 - nb_1^{n-1} = 0$$

$$b_1 = \frac{n-1}{n} v_1$$

You should see that as the number of bidders increases so do each bidder's optimal bid.