University of California, Berkeley Spring 2007
ECON 100A Section 115, 116

Cost of Production : An Example

What you should get out of this example:

- ✓ Understand the technical derivation of optimal inputs in Cost Minimization
- ✓ Able to show the return to scale of a production function
- ✓ Able to derive various cost functions, including
  - Total cost
  - Average cost
  - Marginal cost
  - Fixed cost
  - Variable cost
- ✓ Able to show that the intersection point between the average cost curve and the marginal cost curve corresponds to the minimum of average cost.
- ✓ Understand the differences between long run and short run. Able to derive the short run production and cost functions.
- ✓ Realize that total cost and average cost for producing a given level of output are weakly lower in the long run than in short run, and that the same does not hold in general for marginal cost.

Important: In this example I put Capital (K) on the horizontal axis; in dealing with problem sets and exams remember to check if the question directs you to put Labor (L) on the horizontal axis instead.

You are a newly hired executive at Orange Computers, Inc., the manufacturer of the highly successful OrangePods. In the production of OrangePods two factors are used—machines (K) and workers (L), according to the production function $F(K,L) = L^{1/2}K^{1/2}$. The current rent for machine is 1 and wage rate is 1. In order to operate in the State of Nowhere OrangePods has to pay a lump-sum registration tax of 1 (million) dollars.

1. What it the total cost, in terms of K and L?

$$\begin{aligned} C(K,L) &= \text{tax} + \text{rent} \times \text{machines} + \text{wage} \times \text{workers} \\ &= 1 + 1 \cdot K + 1 \cdot L \\ &= 1 + K + L \end{aligned}$$

2. Derive the optimal long run combination of inputs for given amount of OrangePods q

This is an exercise of cost minimization.

$$\begin{aligned} F(K,L) &= L^{1/2}K^{1/2} \\ MP_K &= \tfrac{1}{2} L^{1/2} K^{-1/2} \\ MP_L &= \tfrac{1}{2} K^{1/2} L^{-1/2} \\ MRTS &= \frac{MP_K}{MP_L} = \frac{L}{K} \\ \text{input price ratio} &= \frac{r}{w} = 1 \\ MRTS &= \text{input price ratio} \\ \frac{L}{K} &= 1 \\ L &= K \end{aligned}$$

Substituting $L = K$ into $F(K,L) = L^{1/2}K^{1/2}$ gives $K^{LR}(q) = L^{LR}(q) = q$

3. What returns to scale does Orange Computer's production function exhibits?

$$\begin{aligned} F(\alpha K, \alpha L) &= (\alpha L)^{1/2}(\alpha K)^{1/2} \\ &= \alpha^{1/2} L^{1/2} \alpha^{1/2} K^{1/2} \\ &= \alpha L^{1/2} \alpha K^{1/2} \\ &= \alpha F(K, L) \end{aligned}$$

So constant returns to scale.

4. Find the long run total cost, long run fixed cost and long run variable cost. Where does the fixed cost and variable cost come from?

$$\begin{aligned} C^{LR}(q) &= 1 + K^{LR}(q) + L^{LR}(q) \\ &= 1 + q + q \\ &= 1 + 2q \\ FC^{LR} &= C^{LR}(0) = 1 \\ VC^{LR}(q) &= C^{LR}(q) - FC^{LR} = 2q \end{aligned}$$

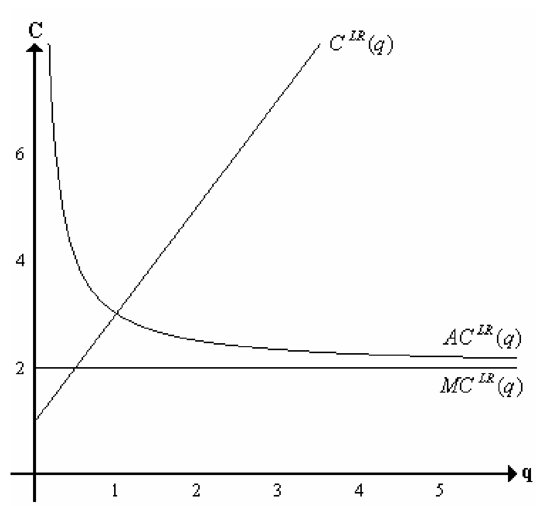
The fixed cost corresponds to the registration tax, while the variable cost corresponds to the wage and rent paid to raise output by 1 unit.

5. What is the long run average cost and long run marginal cost? Verify that their intersection corresponds to the minimum of long run average cost.

$$AC^{LR}(q) = \frac{C^{LR}(q)}{q} = \frac{1}{q} + 2$$

$$MC^{LR}(q) = \frac{dC^{LR}(q)}{dq} = 2$$

Note that $AC^{LR}(q)$ is always decreasing in q (you can take its derivative to verify that) while $MC^{LR}(q)$ is constant. $AC^{LR}(q)$ is always decreasing means that its minimum is at infinity; the intersection point of $AC(q)$ and $MC(q)$ is thus at infinity (so strictly speaking they do not really intersect). We can verify that this is true:



$$\lim_{q\to\infty} AC(q) = \frac{1}{\infty} + 2 = 2 = MC(q) = \lim_{q\to\infty} MC(q)$$

6. Suppose Orange Computers has been producing 9 (hundred thousand) units of OrangePods for quite some time, so that it is using the optimal combination of inputs from above. Demand for OrangePods surged to 12 because it is expected to be out of production soon. It is too late to order new machines so the executives have decided to hire more workers. How many workers are needed?

First find the long run optimal *K* and *L*. From part 2 we know this is $K^{LR} = L^{LR} = 9$.
Now in the short run $K K^{SR}$ is fixed at $K^{LR} = 9$, so

$$\begin{aligned} F^{SR}(L) &= F(9, L) \\ &= 9^{1/2} L^{1/2} \\ &= 3L^{1/2} \end{aligned}$$

then for any given output level q,

$$\begin{aligned} F^{SR}(L) &= q \\ 3L^{1/2} &= q \\ L^{SR}(q) &= \left(\frac{q}{3}\right)^2 \end{aligned}$$

Now we have $q = 12$, so $L = (12/3)^2 = 16$

7. Find the short run total cost, short run fixed cost and short run variable cost. Where does the fixed cost and variable cost come from?

We use back the formula $L^{SR} = \left(\frac{q}{3}\right)^2$ from part 4,

$$\begin{aligned} C^{SR}(q) &= 1 + K^{SR} + L^{SR}(q) \\ &= 1 + 9 + \left(\frac{q}{3}\right)^2 \\ &= 10 + \frac{q^2}{9} \\ FC^{SR} &= C^{SR}(0) = 10 \\ VC^{SR}(q) &= C^{SR}(q) - FC^{SR} = \left(\frac{q}{3}\right)^2 \end{aligned}$$

The fixed cost corresponds to the registration tax and the rent for the 9 units of *K*.
The variable cost corresponds to the wage paid to increase output by one unit.

8. Derive the short run average cost and short run marginal cost. Verify that their intersection corresponds to the minimum of short run average cost.

$$AC^{SR}(q) = \frac{C^{SR}(q)}{q}$$

$$= \frac{10}{q} + \frac{q^2}{9q} = \frac{90 + q^2}{9q}$$

$$MC^{SR}(q) = \frac{dC^{SR}(q)}{dq}$$

$$= \frac{2q}{9}$$

Note that $C^{SR}(q)$ and $AC^{SR}(q)$ are both quadratic. You should be able to see that

$C^{SR}(q)$ reaches its minimum at q = 0, since $\left(\frac{q}{3}\right)^2 > 0$ for all $q \neq 0$.

For average cost

$$AC^{SR}(q) = \frac{10}{q} + \frac{q^2}{9q} = \frac{10}{q} + \frac{q}{9} = \infty \quad \text{if} \quad q = 0 \quad \text{or} \quad q = \infty$$

So there is a minimum in the middle. Find this minimum by differentiation,

$$\frac{dAC^{SR}(q)}{dq} = 0$$

$$-\frac{10}{q^2} + \frac{1}{9} = 0$$

$$q = \sqrt{90}$$

On the other hand the intersection point of $AC^{SR}(q)$ and $MC^{SR}(q)$ is

$$AC^{SR}(q) = MC^{SR}(q)$$

$$\frac{10}{q} + \frac{q}{9} = \frac{2q}{9}$$

$$q^2 = 90$$

$$q = \sqrt{90}$$

So the intersection point of short run average cost and short run marginal cost does corresponds to the minimum of short run average cost.

9. Steven, the CEO of Orange, plans to stay with the combination of inputs in part 4 in the future. Why would you think this is not the best idea on Earth? What would be a better plan?

From part 6 the short run total cost in producing 12 units of output is

$$C^{SR}(12) = 10 + (12/3)^2 = 26$$

While the long run total cost is, referring back to part 3,

$$C^{LR}(12) = 1 + 2(12) = 25$$

Since $C^{LR}(12) < C^{SR}(12)$ it is better for the company to use the long run combination of inputs in the future. From part 2 we know this is $(K, L) = (25,25)$.

In general since the company can adjust all inputs (in this example K and L) in the long run, the long run total cost is always weakly lower (i.e. either lower or equal) than the short run total cost—the company can always choose the short run input values in the long run if it wants to. Since $AC = TC/q$ this is also true for average cost. This is, however, in general not true for marginal cost. Intuitively when the company has so much input already in place, producing small levels of output could cost very little marginally in the short run.

10. Verify that the long run total cost curve, average cost curve and marginal cost curve intersect with their short run counterparts at a certain point. What is the rationale behind this result?

$$C^{LR}(q) = C^{SR}(q)$$
$$1 + 2q = 10 + \frac{q^2}{9}$$
$$q^2 - 18q + 81 = 0$$
$$(q-9)^2 = 0$$
$$q = 9$$

$$MC^{LR}(q) = MC^{SR}(q)$$
$$2 = \frac{2q}{9}$$
$$q = 9$$

$$AC^{LR}(q) = AC^{SR}(q)$$
$$\frac{1}{q} + 2 = \frac{10}{q} + \frac{q}{9}$$
$$q^2 + 2q + 81 = 0$$
$$q = 9$$

So the three sets of cost curves all intersect at $q = 9$. The rationale is that we get the short run production function by fixing the short run K to be the optimal long run K for producing 9 units of output; so when $q = 9$ the short run production function is identical to the long run production function. Since the production functions are identical the cost functions have to be identical at $q = 9$.

The Various Cost Functions